PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

On the parametric rolling of ships in a following sea under simultaneous nonlinear periodic surging

K. J. Spyrou

Phil. Trans. R. Soc. Lond. A 2000 **358**, 1813-1834
doi: 10.1098/rsta.2000.0616

THE ROYAL SOCIETY

On the parametric rolling of ships in a following sea under simultaneous nonlinear periodic surging

By K. J. Spyrou

Department of Naval Architecture and Marine Engineering, National Technical University of Athens, 9 Iroon Polytechniou, Zographou, Athens 157 73, Greece

A new approach to the study of the parametric rolling of ships in a following sea is presented. The new aspect is the consideration of the interference of surge with the roll dynamics. When the waves are long and steep, the oscillatory component of the surge velocity can become large when compared with the mean value. As the oscillatory surge grows in amplitude, it tends also to become asymmetric due to the existence of nonlinearity. The nature of asymmetric surging is such that a ship spends more time on the crests than on the troughs of the waves. This means, however, that the probability of capsize is increased because a ship's roll-restoring capability around the crest is at a minimum. We propose a new second-order differential equation of roll which incorporates automatically the surge effect through appropriate position-dependent coefficients. We explore numerically how this asymmetry in surge influences the build up of parametric rolling. The layout of the stability transition lines of the coupled system was found to be notably different from that of a Mathieu-type system. We pay attention also to the vicinity of surf-riding, where the capsize is more of a 'pure-loss' type.

Keywords: ship; capsize; parametric rolling; roll; surge; nonlinear dynamics

1. Introduction

It is well known that in a following seaway a ship may become unstable due to an intensively fluctuating roll 'righting arm', primarily the result of a variation of the submerged part of the hull between the crests and the troughs of the waves (Grim 1952). Commonly, mathematical models with a time-dependent restoring term, in many cases as simple as a Mathieu equation with damping, are employed for investigating the dynamic behaviour of a ship subjected to such an 'internal forcing' effect from the waves (Kerwin 1955). In more recent years, the nonlinear terms in restoring and in damping are included in these models so that roll behaviour far from equilibrium, and especially the occurrence of capsize, can be investigated (Blocki 1980; Sanchez & Nayfeh 1990; Kan 1992). A detailed list of references with various earlier approaches (covering deterministic or stochastic excitation, single-degree or coupled models, etc.) is provided in Spyrou (2000).

Since a numerical treatment of the roll equation is not constrained in any sense by the consideration of steep waves, large amplitudes of the fluctuating ('parametric') terms are easily taken into account. However, the use of a mathematical model with a time-dependent restoring should not be overemphasized because it is, in fact, devoid of a true physical meaning (restoring is, of course, position dependent).

Consider a ship moving in the same direction with a steep sinusoidal wave: the roll-righting arm will be some periodic function of $k\xi - \omega t$. The conventional notation is applied here with k being the wavenumber, ξ the distance in the direction of wave propagation of the ship's origin from a fixed datum, ω the wave frequency and t time. Since $\xi = ct + x$ and $\omega = kc$, we may write $k\xi - \omega t = kx$, where x is the position of the ship's origin with respect to a 'wave' system, fixed on a trough and thus moving with the wave celerity c. In a following sea, the frequency of encounter is $\omega_\mathrm{e} = k(c - u)$, where u is the speed. If we multiply both sides by t, we obtain $\omega_\mathrm{e} t = k(ct - ut)$. For constant u, the relative position can be expressed as $x = (u - c)t$, which leads further to the relation $\omega_\mathrm{e} t = -kx$. The representations of variable restoring based on kx and $\omega_\mathrm{e} t$ are therefore interchangeable (or, in other words, the Mathieu-type model is valid) when the speed is nearly constant, but not in other cases. Equivalence entails the periodic surging due to the waves to be small, a requirement practically satisfied when the waves are not steep. Naturally, the question arises as to how a significant periodic surging could influence the onset of instability and, more importantly, whether capsize would be rendered more likely.

The assumption of constant forward speed 'contains' a rather fundamental inconsistency: in the long and steep waves where significant roll-righting-arm reductions occur, the fluctuating part of surge velocity is likely to build up to become a large percentage of the ship's still-water speed. Definitive experimental evidence about this was published by Kan (1990). Another important feature which becomes increasingly relevant as the surge motion grows in amplitude is that, in a sinusoidal wave, the wave surge force will also be a sinusoidal (and thus nonlinear) function of ship position. The nature of this effect is such that a ship will be spending more time around the crests than around the troughs. Experiments based on free-running models in waves have confirmed this behaviour (see, for example, Grochowalski 1989; Kan 1990). The net effect of surge nonlinearity is therefore that a ship stays longer in the area of the wave where it is most vulnerable. If, for example, roll restoring around the crest is negative, more time for divergence from the upright state will be available and capsize should become more likely.

Although the asymmetric surging and the Mathieu-type instability of roll, both phenomena of the following sea, have been repeatedly examined in the past, they are always considered independently and for this reason their combined effect remains unknown. A very interesting aspect, for example, is how this interference of surge with roll will affect the stability transition curves of the trivial 'upright' state in comparison with a conventional Mathieu-type system; and ultimately, what will be the effect, in quantitative terms, on a ship's propensity towards capsize. Answers to these questions will be sought in the present paper. Moreover, we shall explore the possibility of developing a new second-order differential equation of roll which automatically takes into account the coupling effect with surge. This may have limited immediate use at this stage, given the easier option of a direct integration of the motion equations. However, this new roll equation might inspire some further analytical work, which could lead to a closed-form design formula for capsize in following seas.

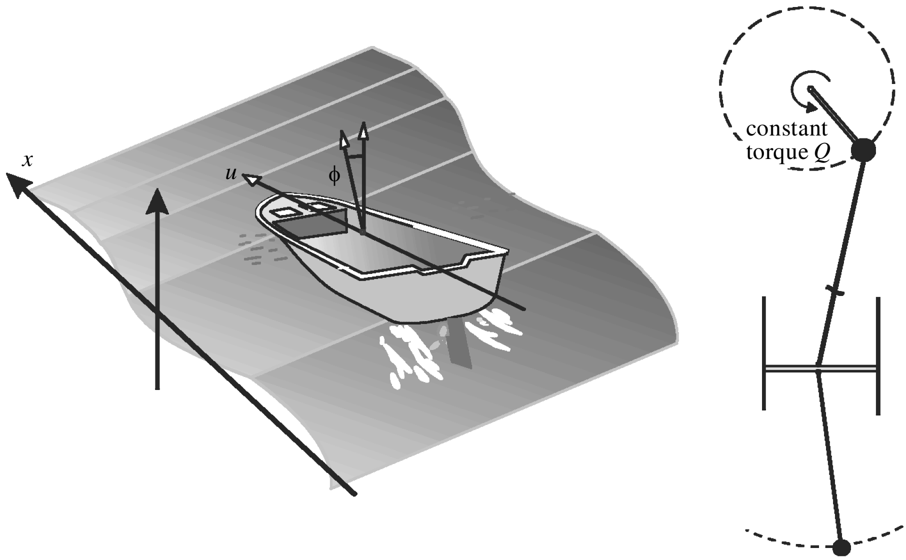

Figure 1. A simple mechanical analogue based on coupled penduli.

We should note that our results are likely to be of interest to a wider audience. At a fundamental level, the problem which we are trying to solve is based on the behaviour of a pendulum forced parametrically through coupling with another pendulum performing full rotations under the action of some constant external torque (figure 1).

2. Nature of surge nonlinearity and character of the response

Consider the balance of forces along the longitudinal axis of a ship travelling in long following waves: in order to sustain the forward motion, the thrust generated by the propeller should counteract the inertial and drag force, plus an alternating position-dependent wave force. The latter tends to push the ship forward when the middle of the ship lies on the downslope, while the forward motion is resisted when the ship lies on the upslope. We assume also that the total resistance of the ship may be split into two parts: a time-independent still-water resistance component and a periodic component due to the existence of the waves, which are assumed to be at least as long as the ship. Considering a simple sinusoidal wave, the application of Newton's second law for surge leads to the following equation of motion:

$$(m - X_{\dot{u}})\dot{u} + [R(u) - T(u,n)] + f\sin(kx) = 0, \tag{2.1}$$

where m, $-X_{\dot{u}}$ are, respectively, the ship's mass and the added mass of surge, R is the resistance force in still water, T is the propeller thrust (assumed unaffected by the wave) and f is the amplitude of the surge wave force. The dot indicates differentiation with respect to the real time t. Although later we shall consider also the roll angle φ, the assumption is made that the effect of roll on the various force components of (2.1) is negligible.

Resistance will be some function of velocity u and may be approximated by a polynomial without constant term like the following:

$$R(u) = r_1 u + r_2 u^2 + r_3 u^3, \tag{2.2}$$

where r_i, $i = 1, 2, 3$, are appropriate coefficients. Likewise, and in accordance with standard practice, the propeller thrust is expressed in terms of the thrust coefficient, K_{T},

$$T(u; n) = (1 - t_{\mathrm{p}})\rho n^2 D_{\mathrm{p}}^4 K_{\mathrm{T}}(u; n), \tag{2.3}$$

where t_{p} is the thrust-deduction coefficient, ρ is water density, n is propeller rate and D_{p} is the propeller diameter. The thrust coefficient K_{T} is written further as a second-order polynomial of the speed of advance $J(u; n)$,

$$K_{\mathrm{T}}(u; n) = \kappa_0 + \kappa_1 J(u; n) + \kappa_2 J^2(u; n), \tag{2.4}$$

with

$$J(u; n) = \frac{u(1 - w_{\mathrm{p}})}{n D_{\mathrm{p}}}, \tag{2.5}$$

where κ_i, $i = 0, 1, 2$, are appropriate coefficients and w_{p} is the wake fraction. Substitutions of (2.5) into (2.4) and then of (2.4) into (2.3) yield

$$T(u; n) = \tau_2 u^2 + \tau_1 u n + \tau_0 n^2, \tag{2.6}$$

where

$$\tau_2 = \kappa_2(1 - t_{\mathrm{p}})(1 - w_{\mathrm{p}})^2 \rho D_{\mathrm{p}}^2, \quad \tau_1 = \kappa_1(1 - t_{\mathrm{p}})(1 - w_{\mathrm{p}})\rho D_{\mathrm{p}}^3, \quad \tau_0 = \kappa_0(1 - t_{\mathrm{p}})\rho D_{\mathrm{p}}^4.$$

Instead of measuring the surge velocity with respect to an observer fixed on Earth, it is more convenient to measure it relative to the wave celerity. The relative velocity $\dot{x}$ will be expressed then as $\dot{x} = u - c$. By substituting (2.2) and (2.6) into (2.1), and then expressing everything in terms of $\dot{x}$, rather than in terms of u, we obtain, finally, the following second-order differential equation for x:

$$\begin{aligned} (m - X_{\dot{u}})\ddot{x} &+ \{[3r_3 c^2 + 2(r_2 - \tau_0)c + r_1] - \tau_1 n\}\dot{x} \\ &+ [3r_3 c + (r_2 - \tau_0)]\dot{x}^2 + r_3 \dot{x}^3 + f \sin(kx) \\ &= \underbrace{(\tau_0 c^2 + \tau_1 c n + \tau_2 n^2)}_{T(c;n)} - \underbrace{(r_1 c + r_2 c^2 + r_3 c^3)}_{R(c)}. \end{aligned} \tag{2.7}$$

The above is basically the equation of a pendulum with nonlinear damping, forced by a constant external torque. If the amplitude of the wave force in the surge direction is small, then the stiffness term of (2.7) becomes almost unimportant. Combined with the fact that damping's nonlinearity is not strong, behaviour will be basically linear. Even when zero-encounter frequency is approached, nothing extraordinary would arise (the hydromechanics will change if the Froude number becomes very high and the ship enters a pre-planing or planing mode where it experiences an upward dynamic lift; but we shall not be concerned with this side of the problem in the present analysis). If the waves become steeper, however, the nonlinear nature of the pendulum-like surge equation (2.7) will start playing an increasingly dominant role.

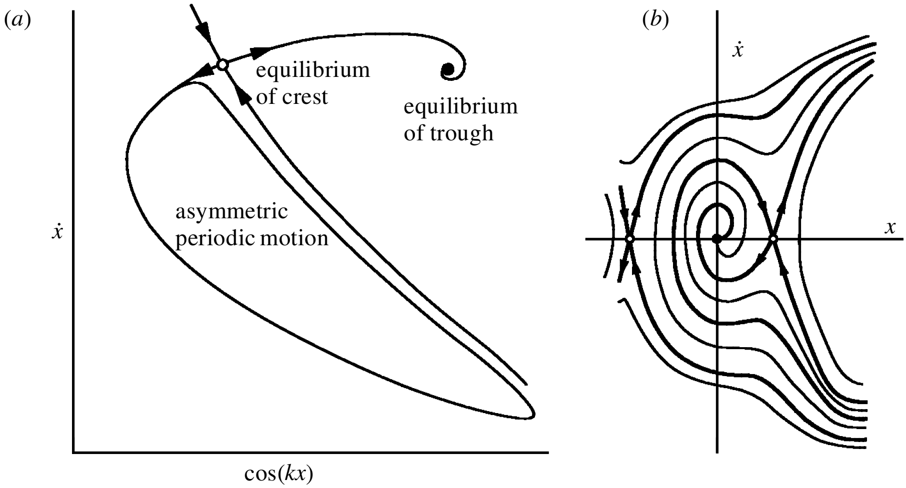

Figure 2. Arrangement of the asymmetric periodic surge and the two coexisting stationary solutions. (*a*) On the phase plane $[\cos(kx), \dot{x}]$. (*b*) Qualitative representation of the 'flow' on the plane $[x, \dot{x}]$.

A very notable characteristic of the response in that wave regime is that higher-order harmonics will appear and the response tends to become increasingly asymmetric. Moreover, a pair of fixed points representing the condition, known in the literature as *surf-riding*, starts to coexist with the periodic motion. On the basis of the analogy with the pendulum, such a development may be understood fairly easily. Depending on the initial state and the magnitude of the constant external torque, a pendulum would either perform full rotations or it would stay at an asymmetric equilibrium position. In fact, such equilibria come into existence in pairs and they are located symmetrically around the horizontal line passing from the centre of the pendulum (or around the node of the wave's upslope, considering the ship dynamics). However, only the lower equilibrium (corresponding to a location nearer to the trough) is stable. The inset of the unstable point (located nearer to the crest) separates the basin of the stable periodic state from that of the stationary state.

The underlying cause of the asymmetry in the periodic surging motion is, as is usual in nonlinear oscillations, an interaction phenomenon. This is depicted nicely on the basis of the phase-plane $[\cos(kx), u]$ (figure 2).† As the steady periodic orbit approaches the saddle point near the crest, the character of the periodic response is increasingly determined by the distance from the manifolds of this saddle.

The critical combination of external forces which would create surf-riding equilibria can be derived on the basis of equation (2.7). If $-1 \leqslant (T(c;n) - R(c))/f \leqslant 1$, then $-1 \leqslant \sin kx \leqslant 1$, and therefore two families of equilibria become possible, located at

$$x = \frac{2\nu\pi}{k} + \frac{1}{k}\arcsin\left[\frac{T(c;n) - R(c)}{f}\right],$$

† A cyclic function of ship position is used so that the steady-state orbit is contained in the range $[0, 1]$. This should not obscure the fact, however, that the character of the motion is likely to be non-cyclic. The undamped version of (2.7) receives an elliptic solution. The modulus of this solution will vary as the saddle inset is approached (by increasing the wave steepness or the propeller rate).

$$x = \frac{(2\nu+1)\pi}{k} - \frac{1}{k}\arcsin\left[\frac{T(c;n)-R(c)}{f}\right],$$

where ν is any integer.

A similar sequence of phenomena would have been realized if the propeller rate, rather than the wave steepness, had been selected as the control parameter (but the wave steepness should be kept fixed at some relatively high value). With respect to (2.7), this means stepping up the torque while lowering the linear damping. No matter which is the control parameter, the oscillatory-type response is destined to disappear altogether at a *homoclinic saddle connection* event. This is a well-known phenomenon of global bifurcation resulting from a collision of a limit cycle with a saddle point in state space (Spyrou 1996). A remarkable and important-for-safety characteristic of this event is that it occurs at a nominal speed which is much lower than the wave celerity.†

With a very considerable increase of speed, the ship should eventually move out of the surf-riding regime and return to periodic-type motions, this time overtaking the waves. For ordinary displacement-type ships, however, such a scenario very rarely represents a practical option.

3. A new equation of coupled roll

We consider the differential equation of roll motion in an exactly following sea,

$$(I - K_{\ddot{\varphi}})\ddot{\varphi} + D(\dot{\varphi}) + C(\varphi, x) = 0. \tag{3.1}$$

In (3.1), φ is the roll angle, I is the roll moment of inertia and $-K_{\ddot{\varphi}}$ is the 'added' moment, $D(\dot{\varphi})$ is the roll damping function and $C(\varphi, x)$ is the position-dependent roll restoring. We have assumed that the roll damping is not influenced significantly by the wave, in which case $D(\dot{\varphi})$ may be expressed in the customary manner on the basis of a linear plus a cubic (or a linear plus an absolute quadratic) roll velocity term. A realistic generic expression for restoring is not easy to determine due to the combination of a strong nonlinearity with position dependence. The simplest and most common approach is to consider the time dependence only in the linear restoring term which, however, has no physical basis.

By dividing (3.1) with the moment of inertia and then normalizing the roll angle φ on the basis of the angle of 'vanishing stability' φ_{v}, equation (3.1) will become

$$\ddot{z} + b_1\dot{z} + b_3\dot{z}^3 + g(z, x) = 0, \tag{3.2}$$

where a damping function with linear and cubic term was adopted, with respective coefficients b_1 and b_3. The position-dependent restoring function $g(z, x)$ is not determined explicitly at this stage.

Given an initial ship state, the time evolution of z for a specific combination of propeller rate and wave characteristics may be found by solving the equation of surge (2.7) simultaneously with the equation of roll (3.2). The phase-space dimension changes, however, from three $(z, \dot{z}, t)$ to five, due to the extra pair $(x, \dot{x})$. The

† For a purse-seiner vessel whose behaviour was investigated in waves with $\lambda/L = 2.0$, $H/\lambda = \frac{1}{20}$, the homoclinic connection occurred at $Fr = 0.402$. This is *ca.* 71% of the celerity value (0.564). It is notable also that due to the fact that surf-riding equilibria exist for $Fr > 0.324$, the transition to surf-riding can take place at a nominal Froude number which is only 57% of the celerity value.

direct method for studying the behaviour of such a system is by solving the coupled equations of motion numerically (see § 4 below). We might also think of identifying an approximate solution $x(t)$ (essentially the rotational motion of the pendulum) from the surge equation on the basis of some perturbation method and then introduce this solution into the roll equation. The problem with this approach, however, is that in the parameter region of our interest the solution is strongly elliptic. Moreover, the value of damping in the surge mode is very high. Thus the hope for success is limited.

Due to the specific form of the surge equation, a third option is also available. It will be shown that it is possible to produce a new roll equation, without using a perturbation-like approach, by combining equations (2.7) and (3.2). We are required, however, to change the variable with respect to which the differentiation is carried out. Instead of using the time variable t, we shall express everything in terms of the relative position x (differentiation with respect to x will be indicated by a prime). The following transformations are applied:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = z'\dot{x}, \tag{3.3}$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \mathrm{d}\left[\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)\right]\Big/\mathrm{d}t = \left(\frac{\mathrm{d}^2 z}{\mathrm{d}x\,\mathrm{d}t}\right)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\right). \tag{3.4}$$

Substitution of $\mathrm{d}z/\mathrm{d}t$ from (3.3) into (3.4) yields

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}\right)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\right) = z''\dot{x}^2 + z'\ddot{x}. \tag{3.5}$$

Let us introduce further equations (3.3) and (3.5) into (3.2):

$$\dot{x}^2 z'' + [\ddot{x}z' + b_1\dot{x}z' + b_3(\dot{x}z')^3] + g(z,x) = 0. \tag{3.6}$$

Consider once more equation (2.7) which, in principle, is not solvable analytically. Had damping been a quadratic function of $\dot{x}$, however, equation (2.7) would essentially be a very special form of the equation of the pendulum for which exact analytical expressions of its phase trajectories (given some initial conditions) can be obtained in the form $\dot{x} = F(x)$ (Stoker 1950). In order to exploit this possibility, we must approximate the third-order damping polynomial by the single quadratic $\gamma(c;n)\dot{x}|\dot{x}|$, whose coefficient γ must be identified. This approximation is legitimate because the graphical forms of the two functions are quite alike.

Consider the damping component $D(\dot{x})$ of (2.7) and the approximate one, say $D_1(\dot{x})$, based on the single quadratic

$$D(\dot{x}) = \underbrace{\{[3r_3c^2 + 2(r_2 - \tau_0)c + r_1] - \tau_1 n\}}_{A_1}\dot{x} + \underbrace{[3r_3c + (r_2 - \tau_0)]}_{A_2}\dot{x}^2 + \underbrace{r_3}_{A_3}\dot{x}^3, \tag{3.7 a}$$

$$D_1(\dot{x}) = \gamma\dot{x}|\dot{x}|. \tag{3.7 b}$$

The coefficient γ should be identified by minimizing the distance between $D(\dot{x})$ and $D_1(\dot{x})$.† More specifically, we want to minimize $S = D(\dot{x}) - D_1(\dot{x})$ over a velocity range determined by lower and upper bounds, respectively, $\dot{x}_1$ and $\dot{x}_2$.

† The calculation of an equivalent quadratic damping should be done, in principle, on the basis of an energy argument, by requesting the energy loss in a 'cycle' to be the same between the two systems. However, since an analytical expression of the solution cannot be obtained, such a method is not useful in this particular context.

A rather straightforward method is to consider a least-squares fit of $D_1(\dot{x})$ on $D(\dot{x})$. By considering l points in the range $[\dot{x}_1, \dot{x}_2]$, we identify the value of γ that minimizes the quantity

$$S = \left(\sum_{i=1}^{l} A_1\dot{x}_i + A_2\dot{x}_i^2 + A_3\dot{x}_i^3 - \gamma\dot{x}_i|\dot{x}_i|\right)^2.$$

This value is given by

$$\gamma = -\left(A_1\sum_{i=1}^{l}\dot{x}_i^3 + A_2\sum_{i=1}^{l}\dot{x}_i^4 + A_3\sum_{i=1}^{l}\dot{x}_i^5\right)\Bigg/\sum_{i=1}^{l}\dot{x}_i^4. \tag{3.8}$$

It is obvious that γ depends on the range of $\dot{x}$. The determination of appropriate lower and upper bounds for $\dot{x}$ is discussed in the appendix.

By introducing the quadratic damping term into (2.7), the following 'equivalent' equation of surge is derived:

$$(m - X_{\dot{u}})\ddot{x} + \gamma(c;n)\dot{x}|\dot{x}| + f\sin(kx) = T(c;n) - R(c). \tag{3.9}$$

On (3.9) are applied the following transformations:

$$y = kx, \qquad p = \frac{\gamma}{k(m - X_{\dot{u}})}, \qquad q = \frac{fk}{(m - X_{\dot{u}})},$$
$$r = \frac{[T(c;n) - R(c)]k}{(m - X_{\dot{u}})}, \qquad \tau = \sqrt{q}t, \qquad v = \frac{\mathrm{d}y}{\mathrm{d}\tau}.$$

Then (3.7) can be recast as follows:

$$v\frac{\mathrm{d}v}{\mathrm{d}y} + pv|v| + \sin y = \frac{r}{q}. \tag{3.10}$$

This may be written further as

$$\frac{\mathrm{d}(v^2)}{\mathrm{d}y} + 2pv|v| = -2\sin y + \frac{2r}{q}, \tag{3.11}$$

or

$$\frac{\mathrm{d}(v^2)}{\mathrm{d}y} + 2pv^2 = -2\sin y + \frac{2r}{q} \quad \text{when } v \geqslant 0, \tag{3.12 a}$$
$$\frac{\mathrm{d}(v^2)}{\mathrm{d}y} - 2pv^2 = -2\sin y + \frac{2r}{q} \quad \text{when } v < 0. \tag{3.12 b}$$

Equations (3.12) obtain the following exact solutions, respectively:

$$v = \sqrt{c_1\mathrm{e}^{-2py} + \frac{2(\cos y - 2p\sin y)}{(1 + 4p^2)} + \frac{r}{pq}} \quad \text{when } v \geqslant 0, \tag{3.13 a}$$
$$v = -\sqrt{c_2\mathrm{e}^{2py} + \frac{2(\cos y + 2p\sin y)}{(1 + 4p^2)} - \frac{r}{pq}} \quad \text{when } v < 0. \tag{3.13 b}$$

The coefficients c_1 and c_2 should be determined on the basis of the initial conditions. We shall go back now to the original variables and parameters and we shall express the solution in terms of the relative position x and real time t:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = \frac{1}{k}\sqrt{c_1 q \mathrm{e}^{-2pkx} + \frac{2q(\cos kx - 2p\sin kx)}{(1+4p^2)} + \frac{r}{p}} \quad \text{for } \dot{x} \geqslant 0, \tag{3.14 a}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = -\frac{1}{k}\sqrt{c_2 q \mathrm{e}^{2pkx} + \frac{2q(\cos kx + 2p\sin kx)}{(1+4p^2)} - \frac{r}{p}} \quad \text{for } \dot{x} < 0. \tag{3.14 b}$$

The second time derivative of x should be

$$\ddot{x} = -pk\dot{x}|\dot{x}| - \frac{q}{k}\sin kx + \frac{r}{k}. \tag{3.15}$$

Through substitution of (3.14) and (3.15) into (3.6), the explicit time dependence is removed completely from the roll equation. Rewriting, for example, (3.2) with linearized still-water restoring, characterized by a natural frequency ω_0, and a linearized damping $2\mu\dot{z}$ (so that direct comparisons with the customary Mathieu-type roll model are possible), we obtain, after the substitutions into (3.6) have been carried out,

$$\begin{aligned}
&\frac{1}{k^2}\left[c_1 q \mathrm{e}^{-2pkx} + \frac{2q}{(1+4p^2)}(\cos kx - 2p\sin kx) + \frac{r}{p}\right] z'' \\
&\quad + \left[-\frac{p}{k}\left(c_1 q \mathrm{e}^{-2pkx} + \frac{2q}{(1+4p^2)}(\cos kx - 2p\sin kx) + \frac{r}{p}\right)\right. \\
&\qquad \left. + \frac{2\mu}{k^2}\sqrt{c_1 q \mathrm{e}^{-2pkx} + \frac{2q}{(1+4p^2)}(\cos kx - 2p\sin kx) + \frac{r}{p}} - \frac{q}{k}\sin kx + \frac{r}{q}\right] z' \\
&\qquad\qquad + \omega_0^2(1 - h\cos kx)z = 0,
\end{aligned} \tag{3.16 a}$$

$$\begin{aligned}
&-\frac{1}{k^2}\left[c_2 q \mathrm{e}^{2pkx} + \frac{2q}{(1+4p^2)}(\cos kx + 2p\sin kx) - \frac{r}{p}\right] z'' \\
&\quad + \left[\frac{p}{k}\left(c_2 q \mathrm{e}^{2pkx} + \frac{2q}{(1+4p^2)}(\cos kx + 2p\sin kx) - \frac{r}{p}\right)\right. \\
&\qquad \left. - \frac{2\mu}{k^2}\sqrt{c_2 q \mathrm{e}^{2pkx} + \frac{2q}{(1+4p^2)}(\cos kx + 2p\sin kx) - \frac{r}{p}} - \frac{q}{k}\sin kx + \frac{r}{q}\right] z' \\
&\qquad\qquad + \omega_0^2(1 - h\cos kx)z = 0,
\end{aligned} \tag{3.16 b}$$

representing the cases $\dot{x} \geqslant 0$ and $\dot{x} < 0$, respectively. In (3.16) we assumed sinusoidal variation of restoring, with amplitude h.

If the waves run faster than the ship, the velocity $\dot{x}$ will be lower than zero because $\dot{x}$ represents the difference of the ship's real surge velocity from the wave celerity. There is only one exception to this which, however, is not of particular interest here. During the transient leading to surf-riding, the velocity will exceed the celerity as the orbit is near and almost parallel to the outset of the saddle of crest. For most conventional ships, their maximum still-water speed is usually below the celerity of those waves whose length is likely to cause problems in the sense discussed. Therefore, it is reasonable to concentrate on those cases where $\dot{x}$ is negative since our intention is to examine behaviour in steep and long waves well before the surf-riding range.

A further remark is that as the ship is trailing behind the waves, $x \to -\infty$ and the exponential terms $e^{\pm 2px}$, representing, in fact, the transient part, will vanish. If we could confine our attention to the effect of steady periodic surging on roll, the following simplified form of the roll equation would be obtained:

$$\begin{aligned} -\frac{1}{k^2}\left[\frac{2q}{(1+4p^2)}(\cos kx + 2p\sin kx) - \frac{r}{p}\right]z'' \\ + \left[\frac{p}{k}\left(\frac{2q}{(1+4p^2)}(\cos kx + 2p\sin kx) - \frac{r}{p}\right)\right. \\ \left. - \frac{2\mu}{k^2}\sqrt{\frac{2q}{(1+4p^2)}(\cos kx + 2p\sin kx) - \frac{r}{p}} - \frac{q}{k}\sin kx + \frac{r}{q}\right]z' \\ + \omega_0^2(1 - h\cos kx)z = 0. \quad (3.17) \end{aligned}$$

Thus we have arrived at a new roll equation with the form $A(x)z'' + B(x)z' + K(x)z = 0$ containing the nonlinear surge effects and having periodic coefficients in all three terms: inertia, damping and restoring.

4. Numerical investigation

(a) Selection of control parameter

When periodic surging is taken into account, the frequency of encounter (and subsequently the frequency ratio) will become periodic too and should no longer be treated as a control variable. At first instance, a mean encounter frequency might seem appropriate to play this role. However, given that our interest lies mainly in the regime of strongly non-cyclic surging motion, the identification of such a mean is problematic. It is far easier to base the control parameter on the frequency of encounter ω_e that would be realized if we operated at steady state and in calm sea. Rather than using directly ω_e, however, we have preferred (as is common for parametric systems) the ratio $a = \omega_0^2/\omega_e^2$, with ω_0 representing the roll's natural frequency. It is easily shown that, for a certain wave, a is linked with the nominal Froude number (a representative of steady motion in still water) through the following relationship:

$$Fr = Fr_c - \frac{\lambda}{L}\frac{\omega_0'}{2\pi\sqrt{a}}, \quad (4.1)$$

where

$$Fr_c = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\lambda}{L}}$$

is the Froude number of the wave celerity and

$$\omega_0' = \omega_0\sqrt{L/g}$$

is the non-dimensionalized natural frequency, L is the ship length, λ is the wave length and g is, this time, the acceleration due to gravity. Furthermore, by matching still-water resistance with thrust at the equilibrium condition, the following expression of the required propeller rate n is found:

$$n = \frac{-d_1 + \sqrt{d_1^2 - 4d_2}}{2}, \tag{4.2}$$

where

$$d_1 = \frac{b_2}{b_3} Fr \sqrt{Lg}, \tag{4.3}$$

$$d_2 = \frac{(b_1 - a_2) Lg Fr^2 - a_3 (Lg)^{3/2} Fr^3 - a_1 \sqrt{Lg} Fr}{b_3}. \tag{4.4}$$

(b) Calculation of the surge wave force

The behaviour of a ship on a wave with $\lambda/L = 2.0$ and variable steepness has been considered in detail. In order to ensure that realistic levels of excitation are used, the amplitude f of the surge force is calculated on the basis of a 34.5 m long purse-seiner vessel which has been under extensive investigation recently (Umeda *et al.* 1995). The amplitude f is determined according to the Froude–Krylov assumption, which is believed to be satisfactory as far as the surge direction is concerned.

It is not difficult to envisage that the stability boundaries will undergo transformations if there is going to be any substantial difference between the coupled and the uncoupled systems. In order to understand how the dynamics of the coupled system emerge as the strength of coupling is raised, we treated the surge force as an independent (control) parameter which is gradually increased towards realistic levels. Then we carried out simulations using characteristic fractions of the wave force.

(c) Roll-restoring function

In the calculation of roll restoring a rather more liberal approach was adopted. As the main objective is to find out how the stability chart of a Mathieu roll equation is changed when coupling with surge is introduced, the following assumptions were made.

(a) Restoring is taken to be a simple linear function of the roll angle.

(b) The variation of restoring as a function of the position x is sinusoidal.

(c) The wave steepness H/λ is connected with the amplitude h of the restoring variation according to the following linear law:

$$H/\lambda = \varepsilon h. \tag{4.5}$$

Of course, such a law is somehow artificial, but on the other hand it contains the key principle of the effect. The same approach can be applied without any problem when an exact law is known.

We have assumed further that $\varepsilon = 1/20$. This would mean that negative metacentric height at the crest would appear if a 1/20 wave steepness was exceeded.†

† In a more practical context, of course, a detailed calculation based on the true hull shape should be carried out in order to find out how roll restoring varies as a function of wave length λ and height H, ship position x and, possibly, speed u if the latter is high.

(d) Roll damping

With the precedence of the use of a linear restoring, it was logical to adopt also a linear function for damping. This does not mean, of course, that the roll dynamics should be studied with a linear equation, but only that, at this stage, we are interested in comparing the transition lines with those of a damped Mathieu equation which is linear. The considered roll equation would therefore assume the following form:

$$\ddot{z} + 2\mu\dot{z} + \omega_0^2[1 - h\cos(kx)]z = 0. \tag{4.6}$$

For the first few resonances, the value of damping will be particularly significant in terms of the least required amplitude of forcing. A damping value of $2\mu = 0.117\,\mathrm{s}^{-1}$ extracted from free-roll decay experiments on a 2 m model of the considered ship has been used as the reference value (Hamamoto *et al.* 1995). Simulations were carried out also for artificially lower damping values.

(e) Roll natural frequency

Taking into account (4.1), the vertices of the transition lines for the undamped Mathieu equation are expected to be encountered at the following (nominal) Froude numbers,

$$Fr = Fr_{\mathrm{c}} - \frac{\omega_0'(\lambda/L)}{\pi\eta}, \tag{4.7}$$

where η is the order of the resonance. Sometimes, however, a ship's roll natural frequency is so high that the principal resonance (at ω_{e} around $2\omega_0$) is not realizable (in a following sea, the encounter frequency is bound to be low). Considering once more the ship model, the natural frequency has been measured to be $\omega_0 = 0.84\,\mathrm{s}^{-1}$ (Umeda *et al.* 1995). For such an ω_0, the required Fr at the principal resonance is negative; something, of course, unrealistic. Therefore, only the higher resonances could be encountered in practice.

Our investigation would not have been complete without considering a case where both the principal and the fundamental resonances are present. One case where this happens theoretically is when the natural frequency is half the original natural frequency ($\omega_0 = 0.42\,\mathrm{s}^{-1}$). The two resonances (always for undamped motion) should then be encountered, respectively, at $Fr = 0.062$ and 0.313. We remark that the $Fr = 0.313$ of the fundamental resonance is very near to the threshold Fr where the stationary solutions of surf-riding become possible ($Fr = 0.324$), creating an interesting possibility for interaction.

(f) Initial conditions

(i) Position on the wave

When restoring is positive everywhere on the wave, we initially place the ship's centre at a wave crest. Otherwise, we select that point where negative restoring begins at the downslope of the wave. This point lies at a distance $\lambda(1 - \arccos(1/h)/2\pi)$ from the next trough. As the waves run faster, the ship will immediately enter into the region of negative restoring.

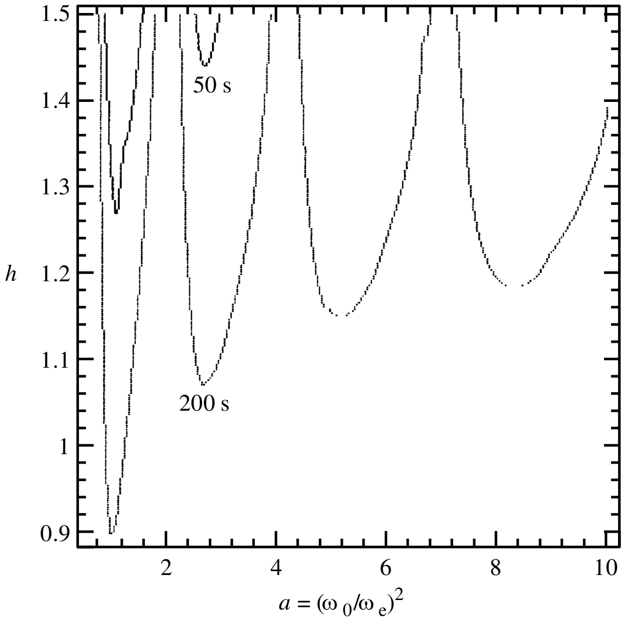

Figure 3. The transition lines for the Mathieu system (no surge coupling) for $t = 200$ s and $t = 50\,\mathrm{s}^{-1}$ (inner lines). The roll damping was $\mu = 0.0585\,\mathrm{s}^{-1}$ and the natural frequency $\omega_0 = 0.84$.

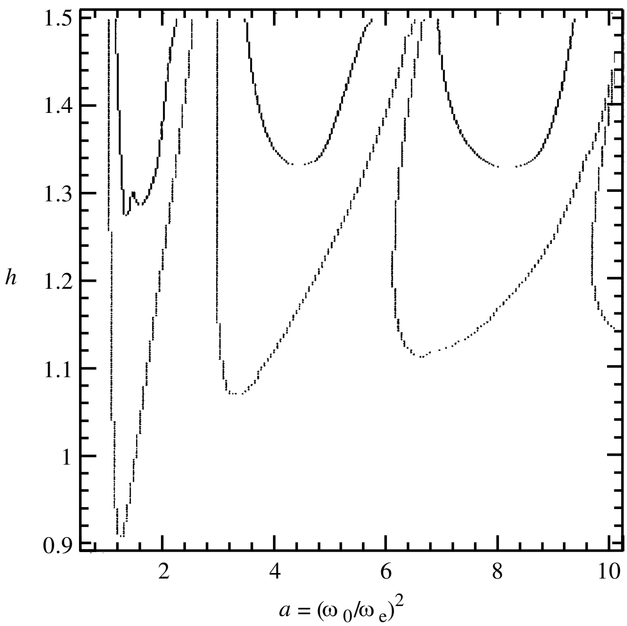

Figure 4. Transition lines for small wave forcing (amplitude $\frac{1}{2}f$). The allowed time and the damping are as in figure 3.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

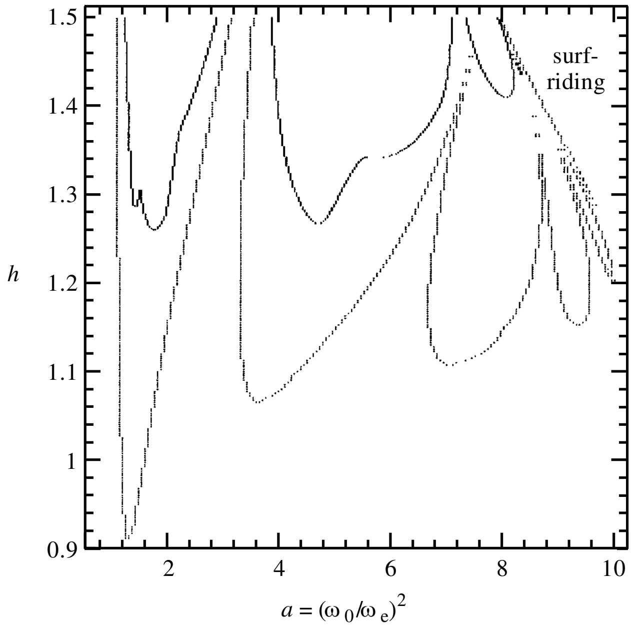

Figure 5. As figure 3, with $\frac{3}{4}f$.

(ii) *Surge velocity*

For the initial surge velocity u_0, we have considered three different possibilities:

(i) $u_0 = u_a$, where u_a is the nominal velocity at the considered value of a;

(ii) $u_0 = c$; and

(iii) $u_0 = 0$.

A high initial velocity, such as the wave celerity, would render surf-riding much more likely in the region just higher than $Fr = 0.324$. However, in that region, we are primarily interested in the asymmetric surging behaviour. The choice of a low speed on the other hand, like $u_0 = 0$, while rendering surf-riding less likely, would also create a rather artificial situation during the first few cycles when the velocity will be in the process of building up. It seemed logical, therefore, to place more emphasis on simulations started with the nominal speed. Even this may not be considered as an ideal solution, because there is still some build up of average velocity in the region of asymmetric surging. The best solution would be to identify the speed at the crest beforehand (by solving the surge equation) and then set this as the initial velocity. This, however, would complicate the simulation process, which was undesirable at this stage.

(iii) *Heel angle*

The initial normalized heel angle is assumed, $z_0 = 0.01$.

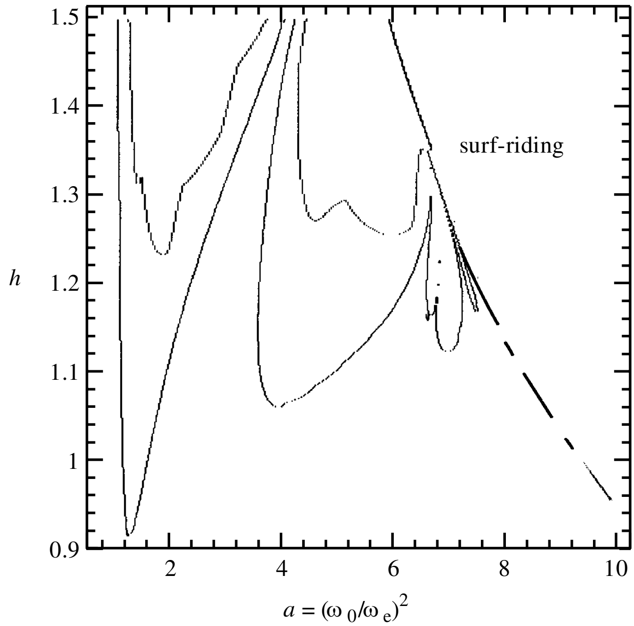

Figure 6. Transition lines with full surge forcing.

(iv) *Roll velocity*

The initial roll velocity $\dot{z}_0$ is zero.

(*g*) *Capsize condition*

When the normalized roll angle had reached the value of 1.0, it was considered that a capsize case had been encountered. Put differently, up to a 100-fold increase of the initial heel has been allowed. Simulations were firstly carried out for 200 s (or until the roll angle had reached the 'capsize' limit $z = 1$, whichever of the two events occurred first). We have investigated the effect of the allowed time by considering also the shorter time duration of 50 s. This is important because the nonlinearity of surge works in such a way that capsize becomes quicker (the Mathieu-based results underpredict the safety margin of the ship). Our interest is focused mainly on the behaviour at low frequencies of encounter where capsize is more likely.

(*h*) *Investigation results*

The transition lines for the Mathieu-type roll equation are shown in figure 3. The result of the introduction of coupling may be seen in figures 4–6 under progressive stepping-up of the wave surge force. Figure 6, especially, shows the transition lines for the full forcing. As said earlier, for the reference roll natural frequency, the principal resonance could not be realized. On the basis of figure 3, which refers to the uncoupled equation, only four resonances are practically significant (the maximum a is 10; for the considered ω_0 that corresponds to $Fr = 0.405$, which is just about the realistic upper limit for displacement ships).

With the introduction of a small surge force, the resonances show a tendency to shift towards higher frequencies while their domains become wider, especially at the

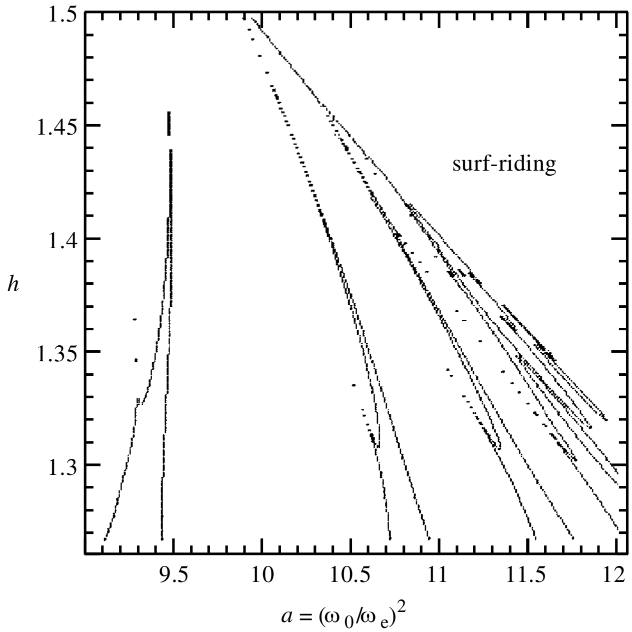

Figure 7. Detail of the surf-riding boundary at $0.625f$.

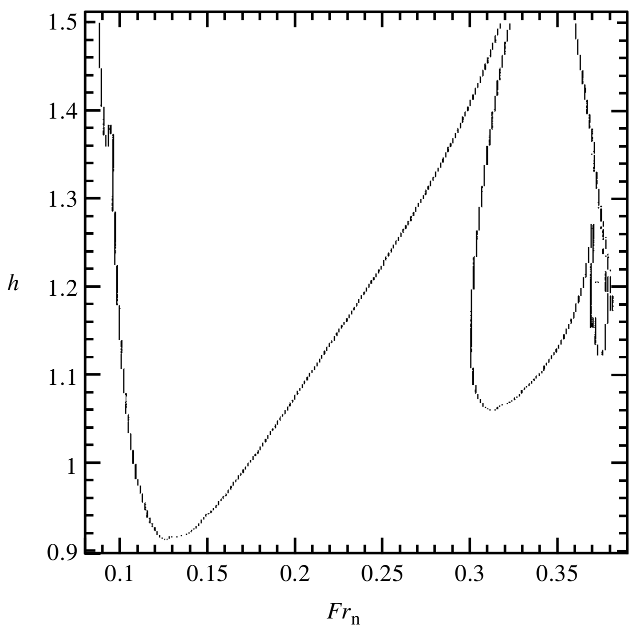

Figure 8. Alternative presentation of the information in figure 6. We have considered on the x-axis the nominal Froude number rather than the ratio of frequencies, since this is more common among naval architects.

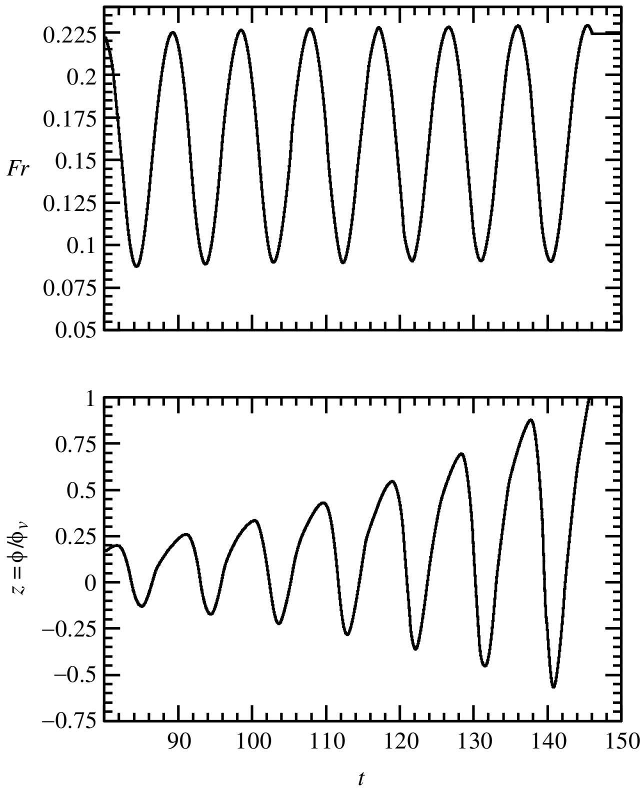

Figure 9. Time record of unstable roll and surge near the 'tip' of the fundamental resonance region ($a = 1.6$, $h = 1.0$).

higher values of h. Although the minimal h required at the first and the second resonance does not seem to change noticeably, the third and the fourth tend to expand towards lower h values. The fourth goes almost out of the considered range of a when the forcing becomes about half the magnitude of the real. At a critical f, a surf-riding domain appears around the upper-right corner of the (a, h) window. A rearrangement of the 'capsize' domain is also taking place. Most remarkably, new smaller 'spikes' clustering from the surf-riding boundary come into existence (figure 7). This mechanism is yet to be understood. A re-plot of figure 6 on the basis of Fr, rather than a, is shown in figure 8.

In figures 3–6 we also show the transition lines for the lower time of 50 s. It is apparent that the effect of the surge coupling is very profound, especially near the encounter frequencies where surf-riding makes its appearance. At $a \approx 6$, for example, the required h has come down to realistic levels.

Some characteristic time-records are presented in figures 9 and 10 (created on the basis of full surge wave forcing). Figure 9 shows the parametric build up of roll near the tip of the fundamental, while figure 10 shows the behaviour near the tip of the immediately higher resonance region. The asymmetry of surge is not present yet as the nominal speed is still low. More interesting, however, is the case of figure 11, which corresponds to a higher value of a. Here, the asymmetry, as well as its effect

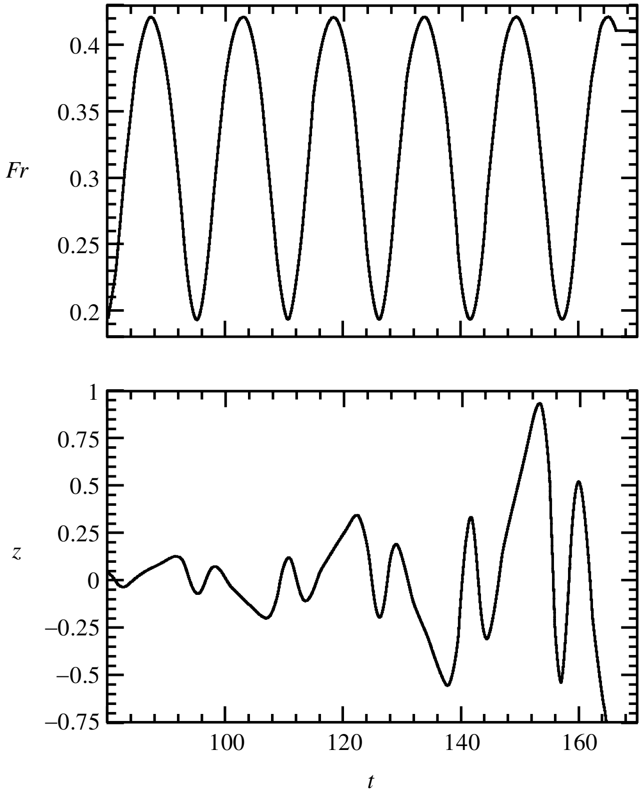

Figure 10. $a = 4.0$, $h = 1.08$.

on roll, have become prevalent. Finally, figure 12 shows the behaviour near the tip of the very narrow resonance region immediately next to the surf-riding boundary.

Reduction of roll damping did not change the layout of the transition lines (figure 13). More interesting results were obtained, however, when the natural frequency was lowered in a way that the principal resonance also became viable (figure 14). The region due to the principal resonance becomes very wide at high values of h. Most notably, a capsize region adjacent to the surf-riding boundary exists which extends to relatively low values of h.

5. Concluding remarks

Further systematic studies will be required in order to assess fully the importance of surge's nonlinearity for capsize. What has been clear so far, however, is that undue disregard of the surge motion leads to serious underestimation of a ship's stability margin. In terms of further development, we should look into the effect of realistic nonlinear (GZ)-curves so that the results obtained can find their way into ship design or operational procedures.

There is no doubt that approaching the problem in a transient sense, i.e. capsize within a given amount of time (which has to be linked rationally with the ship dimensions and speed), is much more relevant to the actual wave environment than the traditional examination of asymptotic stability of steady states (incidentally, it

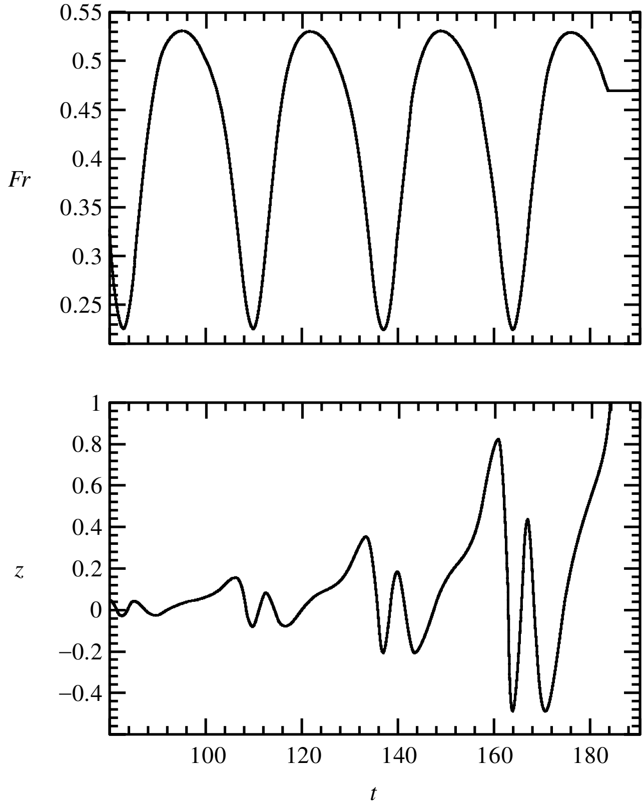

Figure 11. The build up of roll and surge for a combination of a and h where strong asymmetry appears in the surge's record ($a = 7.0$, $h = 1.13$).

is also much easier). Certain concepts should be given more thinking, however, for example, the roll magnification that must be allowed (this will determine the height, in terms of h, of the tips of the transition lines). A solution to this problem may come from the consideration of a fully coupled system, since the initial roll disturbances will be provided through the couplings with the lateral motions. It boils down to the fact that there should be no need for the initial conditions to be artificially determined in advance. We note a recently developed five-degree mathematical model (without surge) for the study of parametric instability (Hamamoto & Munif 1998), which appears to be supplementary to our numerical model.

Appendix A.

In order to obtain an accurate 'least-squares' fit of the damping curve, the lower and the upper bound of the region where asymmetric surging is realized must be defined in a rational way. It seems logical to select as the lower bound that speed where surf-riding from certain initial conditions begins, because thereafter the asymmetry becomes very pronounced. The upper bound should be represented by the speed at which the global bifurcation (homoclinic connection) takes place. This choice cannot be disputed because no periodic response exists after this.

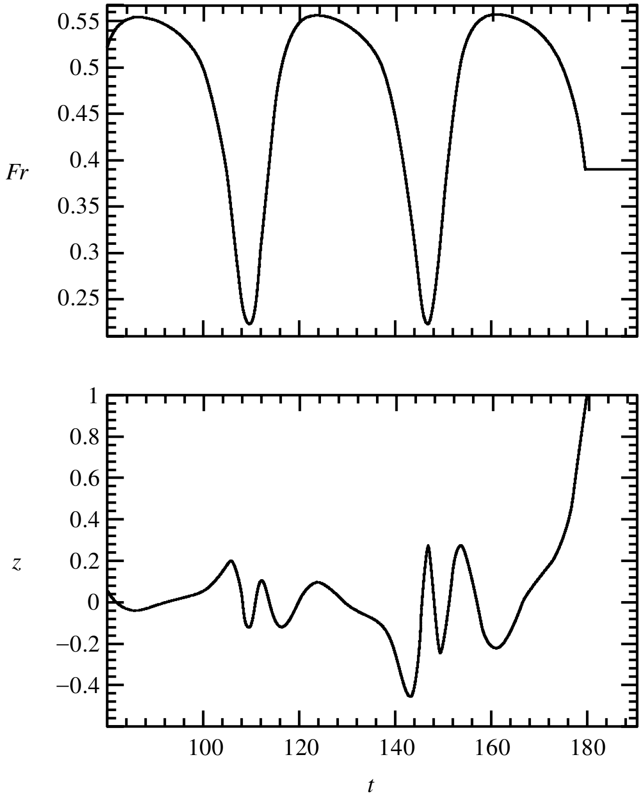

Figure 12. At the 'tip' of the very narrow resonance region adjacent to the surf-riding boundary ($a = 7.5$, $h = 1.17$).

The analytical determination of the lower bound is straightforward. The critical propeller rate is found on the basis of the following relationship:

$$(T(c;n) - R(c))/f = 1,$$

which, after substitution and some rearrangement, leads to the following equation in n:

$$\tau_0 n^2 + \tau_1 cn + [-r_1 c + (\tau_2 - r_2)c^2 - r_3 c^3 - f] = 0.$$

For the acceptable solution, say n_1, we can determine further the corresponding still-water speed u_1 by equating thrust with resistance and solving for u:

$$r_3 u_1^3 + (r_2 - \tau_2)u_1^2 + (r_1 - \tau_1 n_1)u_1 - \tau_0 n^2 = 0.$$

For an easier alternative, we may use the Japanese recommendation (IMO 1991) that surf-riding begins to happen at $V = 1.8\sqrt{L}$, where V is the speed in knots (this is based on 1/10 wave steepness). The corresponding nominal Froude number is 0.296, so a 0.3 Froude number is a realistic lower bound.

Unfortunately, an analytical method for the derivation of the upper speed bound, say u_2, is not known. The condition that needs to be fulfilled is that the unstable fixed point lies on the steady periodic orbit. However, due to the strong ellipticity of the orbit in that regime and the strength of damping, the analytical solution

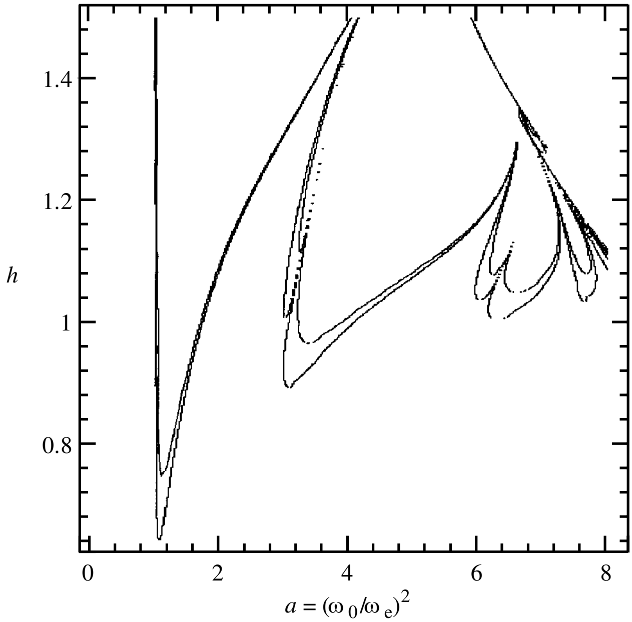

Figure 13. Effect of damping. The curves extending to lower h are drawn with $\frac{1}{4}\mu$ and the others with $\frac{1}{2}\mu$.

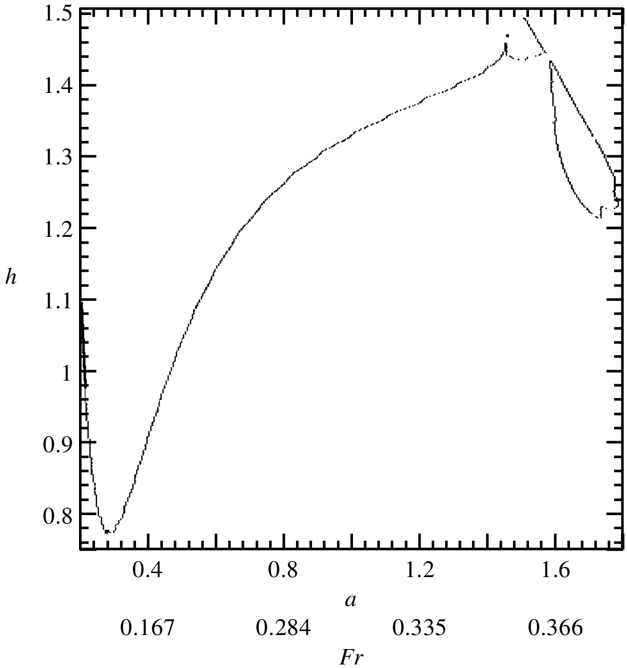

Figure 14. The transition lines for halved natural frequency. The principal resonance becomes possible. Most remarkably, a disconnected capsize region at realistic h values appears adjacent to the surf-riding boundary.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

cannot be obtained. Heuristically (through comparison with experimental results), Kan (1990) proposed that for a surge equation with linearized damping in the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\beta\frac{\mathrm{d}x}{\mathrm{d}t}+\sin x=\alpha$$

the critical α is given by $\alpha=-(4/\pi)/\tanh\beta$.

If we define $u_i=c/m_i$, then $c/m_1<\dot{x}+c<c/m_2$, which further leads to

$$\frac{1-m_1}{m_1}c<\dot{x}<\frac{1-m_2}{m_2}c.$$

By carrying out the least-squares procedure on the basis of the two endpoints only, equation (3.8) will yield the following value of γ:

$$\gamma=\frac{\rho_3A_1+\rho_4cA_2+\rho_5c^2A_3}{\rho_4c}\quad\text{with } \rho_i=\left(\frac{1-m_1}{m_1}\right)^i+\left(\frac{1-m_2}{m_2}\right)^i,\quad i=3,4,5.$$

References

Blocki, W. 1980 Ship safety in connection with parametric resonance of the roll. *Int. Shipbuilding Prog.* **27**, 36–53.

Grim, O. 1952 Rollschwingungen, stabilität und sicherheit im seegang. *Schiffstechnik* **1**, 10–21.

Grochowalski, S. 1989 Investigation into the physics of ship capsizing by combined captive and free-running model tests. *Trans. SNAME* **97**, 169–212.

Hamamoto, M. & Munif, A. 1998 A mathematical model to describe ship motions leading to capsize in severe astern seas. In *Proc. 4th Int. Ship Stability Workshop, St John's, Newfoundland, 27–29 September.*

Hamamoto, M., Umeda, N., Matsuda, A. & Sera, W. 1995 Analyses on low cycle resonance of ships in astern seas. *J. Soc. Naval Architects Japan* **177**, 197–206.

IMO 1991 SLF 36/INF4, submitted by Japan, London.

Kan, M. 1990 Surging of large amplitude and surf-riding of ships in following seas. In *Selected publications in Naval Architecture and Ocean Engineering*, vol. 28. Tokyo: Society of Naval Architects of Japan.

Kan, M. 1992 Chaotic capsizing. In *Proc. 20th ITTC Seakeeping Committee, Osaka, 10–11 September*, pp. 155–180.

Kerwin, J. E. 1955 Notes on rolling in longitudinal waves. *Int. Shipbuilding Prog.* **2**, 597–614.

Sanchez, N. E. & Nayfeh, A. H. 1990 Nonlinear rolling motions of ships in longitudinal waves. *Int. Shipbuilding Prog.* **37**, 247–272.

Spyrou, K. J. 1996 Dynamic instability in quartering seas: the behaviour of a ship during broaching. *J. Ship Res.* **40**, 46–59.

Spyrou, K. J. 2000 Designing against parametric instability in following seas. *Ocean Engng* **27**, 625–653.

Stoker, J. J. 1950 *Nonlinear vibrations in mechanical and electrical systems*. Wiley.

Umeda, N., Hamamoto, M., Takaishi, Y., Chiba, Y., Matsuda, A., Sera, W., Suzuki, S., Spyrou, K. J. & Watanabe, K. 1995 Model experiments of ship capsize in astern seas. *J. Soc. Naval Architects Japan* **177**, 207–218.